Free Vibration of Circular Cylindrical Shells of Finite Length

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Nomenclature

a = mean radius of the shell
A_i,B_i,C_i = arbitrary complex constants
E = modulus of elasticity
h = thickness of shell wall
L = length of shell
m = number of circumferential waves around shell
p = mode identification number; equal to the number of axial nodal circles, excluding the ends, plus 1
t = time
u,v,w = displacements of a point in the middle surface of the shell (axial, tangential, and radial, respectively)
x,ϕ = axial and circumferential coordinates respectively
λ_i = ith root of the auxiliary equation
ρ = mass density of shell
ω = circular frequency
Ω = frequency parameter, $= \rho a^2\omega^2(1 - \nu^2)/E$
ν = poisson's ratio
w' = $\partial w/\partial x$

Introduction

THE problem of calculating the free vibration characteristics of circular cylindrical shells of finite length has been of interest to engineers and scientists for over a century. This interest reaches new heights in today's aerospace industry where the structural response to dynamic loads must be predicted with extreme accuracy.

The equations governing the motion of freely vibrating circular cylindrical shells have evolved from a membrane theory, to a bending theory, to a general theory including both membrane and bending effects. This paper is concerned with determining the free vibration characteristics of circular cylindrical shells of finite length utilizing the general theory. The equations of motion used are those originally derived by Flugge,[1] as uncoupled by Yu.[2] Approximate solutions to these equations have been obtained by imposing various simplifications. These simplifications, however, limit either the range of shell dimensions or specific modes for which the solutions are valid. Smith[3] and Forsberg[4] have presented exact solutions to the general equations; however, Forsberg's solution is valid only for those modes having $m > 1$. Furthermore, Smith and Forsberg have an inherent disadvantage in their method of solution which centers around the treatment of the arbitrary constants in the expressions for the assumed solution, and the roots of the auxiliary equation. These constants and roots are in general complex quantities, and when treated as such lead eventually to the problem of evaluating an eighth order complex determinant. Smith's and Forsberg's treatment requires that these complex constants be replaced by real ones formulated in a specific manner. This process requires a knowledge of the form of the roots of the auxiliary equation prior to actually solving for them. However, once the form of these roots is established, the solution may be written in terms of real arbitrary constants, which then yields the simpler problem of evaluating an eighth-order determinant having only real elements.

Aside from being somewhat laborious, the method used by Smith and Forsberg imposes the restriction that the form of the roots of the auxiliary equation must be known before the real constants can be defined. Should the form of these roots change during the solution process the newly defined constants would no longer be valid. Therefore, it is necessary to continuously monitor the form of these roots throughout the solution process. These disadvantages are overcome in this paper by carrying all complex quantities through the solution process as complex. The eight roots of the auxiliary equation, the arbitrary constants of the assumed solution, and the elements of the determinant resulting from the application of boundary conditions are all found to be complex and are treated as such.

Natural frequencies are presented for the first four radial modes with from one to five circumferential waves for a shell subjected to eight different sets of homogeneous boundary conditions. Comparisons are made to previously determined experimental and theoretical results for two different shells having clamped ends.

Table 1 Boundary conditions[a]

Case	Symbol	Description	Quantities equal to zero at: $x = 0$	$x = L$
1	C-C	Clamped—clamped	u,v,w,w'	u,v,w,w'
2	C-SS	Clamped—simply supported	u,v,w,w'	u,v,w,M_x
3	SS	Simply supported—both ends	u,v,w,M_x	u,v,w,M_x
4	C-FS	Clamped—freely supported	u,v,w,w'	N_x,T_x,w,M_x
5	CNA	Clamped, no axial restraint—both ends	N_x,v,w,w'	N_x,v,w,w'
6	C-F	Clamped—free	u,v,w,w'	N_x,T_x,S_x,M_x
7	SS-F	Simply supported—free	u,v,w,M_x	N_x,T_x,S_x,M_x
8	F-F	Free—free	N_x,T_x,S_x,M_x	N_x,T_x,S_x,M_x

[a] See Fig. 1 for the description of the displacements u, v, and w, and the stress resultants N_x, T_x, S_x, and M_x.

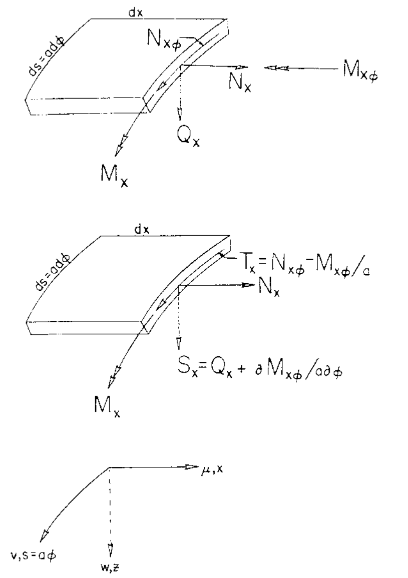

Fig. 1 Displacements and stress resultants.

Solution of the Shell Equations

The vibrations of circular cylindrical shells, as represented by the differential equations derived by Flugge[1] and uncoupled by Yu,[2] may be described by the component displacements u, v, and w (see Fig. 1) of a point on the middle

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Table 2 Natural frequencies (cps) for a clamped—clamped boundary condition[a]

p	m	This[b] paper	Ref. 5 exp.	Ref. 3 exact	Ref. 3 approx.	This[c] paper	Ref. 6 exp.	Ref. 3 exact	Ref. 3 approx.
1	1	3423		3427					
	2	1917		1918		1432	1240	1429	1575
	3	1154	1025	1145	1580	2353	2150	2336	2784
	4	765	700	765	920	4140	3970	4142	4192
	5	581	559	580	641	6497	6320	6500	6565
	6	539	525	538	557	9388	9230	9400	9453
	7	599	587	597	600				
2	1	6412		6423					
	2	3902		3905		2679	2440	2682	3661
	3	2536		2538	4485	2770	2560	2771	3941
	4	1752	1620	1753	2514	4332	4160	4335	5115
	5	1287	1210	1287	1636	6640	6475	6644	6755
	6	1022	980	1022	1189	9518	9380	9480	9580
	7	908	875	907	979				
3	1	8493							
	2	5832		5844					
	3	4051		4054		3564	3380	3570	5870
	4	2918		2921	5008	4711	4540	4717	6167
	5	2190		2192	3190	6897	6720	6903	7110
	6	1719	1650	1720	2244	9737	9540	9630	9796
	7	1431	1395	1431	1705				
4	1	9418							
	2	7299		7303					
	3	5442		5447		4634	4480	4647	8635
	4	4100		4104		5298	5130	5308	7682
	5	3165		3168	5314	7287	7100	7298	7686
	6	2517		2516	3686	10059	9890	9950	10129
	7	2076	1960	2076	2742				

[a] $E = 29.6 \times 10^6$ psi, $\nu = 0.29$, $\rho = 0.733 \times 10^{-3}$ lb-sec²/in.⁴
[b] $L = 12.0$, a = 3.0, h = 0.10, in.
[c] $L = 15.65$, a = 1.924, h = 0.101, in.

surface as follows:

$$u = \sum_{i=1}^{8} A_i \exp(\lambda_i x/a) \cos m\phi \sin\omega t$$

$$v = \sum_{i=1}^{8} B_i \exp(\lambda_i x/a) \sin m\phi \sin\omega t \quad (1)$$

$$w = \sum_{i=1}^{8} C_i \exp(\lambda_i x/a) \cos m\phi \sin\omega t$$

Substitution of Eq. (1) into the differential equations of motion results in the following set of auxiliary equations:

$$A_i\{2\Omega^2/(1-\nu) - (3-\nu)\Omega m^2[1-(\lambda_i/m)^2]/(1-\nu) + m^4[1-(\lambda_i/m)^2]^2\} = C\lambda_i\{2\nu\Omega/(1-\nu) + m^2[1+\nu(\lambda_i/m)^2]\} \quad (2a)$$

$$B_i\{2\Omega^2/(1-\nu) - (3-\nu)\Omega m^2[1-(\lambda_i/m)^2]/(1-\nu) + m^4[1-(\lambda_i/m)^2]^2\} = mC_i\{-2\Omega/(1-\nu) + m^2[1-(2+\nu)(\lambda_i/m)^2]\} \quad (2b)$$

$$\lambda_i^8 + D_1\lambda_i^6 + D_2\lambda_i^4 + D_3\lambda_i^2 + D_4 = 0 \quad (2c)$$

where the coefficients D_j are functions of E, m, h, a, ρ, ν, and ω. Eqs. (2a) and (2b) are linear functions of the complex constants A_i, B_i, and C_i, while Eq. (2c) contains none of these constants but is a polynomial in the λ_i. Eqs. (2a) and (2b) are used to solve for the constants A_i and B_i in terms of the C_i's, so that the displacements of Eq. (1) may be expressed in terms of just these complex constants and the roots of the auxiliary equation, the λ_i.

Application of the Boundary Conditions

Eight different sets of homogeneous boundary conditions are considered herein (see Table 1). Application of a particular set of boundary conditions results in a set of eight homogeneous equations in the eight arbitrary complex constants C_i,

$$[Z]\{C_i\} = \{0\}, i = 1, \ldots 8 \quad (3)$$

For a nontrivial solution of Eq. (3) to exist the determinant of the complex coefficient matrix, $|Z|$, must vanish. This results in the characteristic equation whose eigenvalues determine the natural frequencies of the shell. The corresponding eigenvectors determine the mode shapes.

Results and Conclusions

Frequencies for two different clamped-clamped shells are compared with previously determined experimental and

Table 3 Variation of frequency (cps) with boundary condition[a]

Case symbol		1 C-C	2 C-SS	3 SS	4 C-FS	5 CNA	6 C-F	7 SS-F	8 F-F
p	m								
1	1	3423	3423	3423	1570	2874	1205	1204	4980
	2	1917	1916	1915	1181	1262	479	479	2614
	3	1154	1153	1152	823	656	255	255	1422
	4	765	764	763	566	424	219	218	877
	5	581	580	579	458	380	280	280	633
2	1	6412	6412	6411	4163	6409	3818	3818	6504
	2	3902	3901	3900	2584	3658	2136	2135	4871
	3	2536	2534	2532	1877	2171	1240	1239	3145
	4	1752	1750	1748	1428	1395	800	799	2082
	5	1287	1285	1283	1089	987	597	596	1466
3	1	8493	8496	8491	7089	8493	6817	6818	8840
	2	5832	5839	5839	4728	5786	4406	4405	6601
	3	4051	4049	4047	3200	3878	2824	2822	4762
	4	2918	2916	2913	2401	2673	1906	1904	3423
	5	2190	2187	2184	1896	1929	1371	1369	2515
4	1	9418	9416	9421	9112	9439	9147	9162	9903
	2	7299	7298	7296	6533	7293	6228	6227	7716
	3	5442	5441	5439	4683	5382	4401	4399	6040
	4	4100	4097	4094	3493	3953	3158	3155	4649
	5	3165	3161	3157	2760	2986	2343	2339	3584

[a] All results calculated using the method presented in this paper and the following material properties: $L = 12.0$, $a = 3.0$, $h = 0.010$ in., $E = 29.6 \times 10^6$ psi, $\nu = 0.29$, $\rho = 0.733 \times 10^{-3}$ lb-sec²/in.⁴.

theoretical results in Table 2. These comparisons serve to provide confidence in the validity of the solution. The results obtained using the method presented in this paper are almost identical to the results obtained by Smith.[3] The results shown in Table 2 for the approximate solution were obtained by imposing, on Eq. (2), the condition that $(\lambda_i/m)^2 \ll 1$. The exact solutions are shown to be superior to the approximate solutions, especially for those modes having a small number of circumferential waves. Frequencies, for the first four radial modes having one to five circumferential waves, of a particular shell, subjected to the eight sets of boundary conditions of Table 1, are presented in Table 3.

The advantages of the "exact" solution presented here over those seen previously are: 1) the arbitrary complex constants in the assumed solution need not be redefined as real constants, and 2) the form of the roots of the auxiliary equation need not be monitored during the solution. These advantages yield a saving of computational time and lead to a more truly "general" solution.

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Empirical Analysis of Regional Growth Functions of Turbulent Wakes

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Nomenclature

A_1,A_2 = regression constants
B_1,B_2 = regression coefficients
d = body diameter
M = Mach number
n = number of data points
N = exponent in wake diffusion equations
P = pressure
r = correlation coefficient
Re = Reynolds' number
x,y = physical coordinates
X = x/d
Y = y/d
α = angle of attack

Subscripts

i = ith value of n values
w = wake conditions
∞ = freestream conditions

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Introduction

THE rate of growth of turbulent viscous wakes of axisymmetric bodies at hypersonic velocities has been of interest for some time. Many investigators[1–6] have developed empirical equations describing the growth as a function of distance behind the body. An empirical relationship of the form $Y = A + BX^N$ has been developed where N has a range of values between 0.25 and 0.67. Most of the experimental investigations covered the far wake region ($X > 150$) at hypersonic Mach numbers ($M_\infty > 6$). Also, a theoretical relationship of the form $Y = (A + BX)^N$ with $N \approx 0.33$ has been derived.[7] Assuming the theoretical model is unbounded as a function of velocity and distance behind the body, then experimental evidence at short distances ($X < 25$) behind the body and at lower Mach numbers ($M_\infty < 6$) is not available for comparison with the theoretical model, except for the evidence of Knystautas.[6]

Experimental data in the distance and Mach number regions of $X < 15$ and $M_\infty < 5$ were analyzed for comparison with the theoretical and experimental models. The experimental data were obtained on the U.S. Army Missile Command's aeroballistic range.

The models used were hemisphere cylinders with a base diameter of 0.226 in. and 1.5 calibers in length. The velocities were $M_\infty \approx 4.2$ and $Re \approx 5.38 \times 10^5$. The wake cores were in all cases turbulent up to the throat.

Data Analysis

The interface between the viscid and inviscid portions of the wake was clearly defined in 9 independent photographs and measured on a photoreader machine to obtain coordinated (x_i,y_i) data sets. The data were smoothed by integrating the interface over intervals of 0.5 body diameters and obtaining an average wake width for the interval. The following equations based on the work of other investigators were used in the regression analyses:

$$Y_1 = A_1 + B_1X_1^N \text{ (empirical)} \tag{1}$$

$$Y_2 = (A_2 + B_2X_2)^N \text{ (theoretical)} \tag{2}$$

The analyses allowed N to vary over any real number. For each value of N, the constant A and independent variable coefficient B for the equation was computed. The correlation coefficient between Y and X was also computed to find the highest correlation coefficient. The correlation coefficient which was computed in a numerically systematic manner for all of the data was taken as a measure of the degree of fit between the data and the equation for each value of N. The correlation coefficient is computed as follows:

$$r = \frac{(n\Sigma X_TY_T - \Sigma X_T\Sigma Y_T)}{\{(n\Sigma X_T{}^2 - [\Sigma X_T]^2)(n\Sigma Y_T{}^2 - [\Sigma Y_T]^2)\}^{1/2}} \tag{3}$$

where for Eq. (1)

$$Y = Y_T$$

$$X^N = X_T$$

and for Eq. (2)

$$Y^{1/N} = Y_T$$

$$X = X_T$$

Selection of the analysis giving the highest correlation coefficient for both Eqs. (1) and (2) gives the following:

$$Y = 0.0015 + 0.0221X^{1.44} \; (r = 0.720) \tag{4}$$

$$Y = (0.0123 + 0.1057X)^{2.88} (r = 0.830) \tag{5}$$

These equations are plotted in Fig. 1 with the authors' experimental data. The reason that $Y = 0$ when $X = 0$ in